Equivalence Relations

A relation $\sim$ on a set S is **reflexive** if

$$a \sim a \text{ for all } a \in S.$$

A relation $\sim$ on a set S is **symmetric** if

$$a \sim b \text{ implies } b \sim a \text{ for all } a, b \in S.$$

A relation $\sim$ on a set S is **transitive** if

$$a \sim b \text{ and } b \sim c \text{ implies } a \sim c \text{ for all } a, b, c \in S.$$

A relation $\sim$ on a set S is an **equivalence relation** if $\sim$ is (1) reflexive, (2) symmetric, and (3) transitive.

Example 1. For any set S, $a \sim b$ means $a = b$.

Example 2. $S = \mathbb{Z}$, n is a fixed positive integer,

$$a \sim b \text{ means } a \equiv b \pmod{n}.$$

Example 3. $S = \mathbb{R}$,

$$x \sim y \text{ means } x - y \text{ is an integer.}$$

Example 4. Consider the set of all triangles in $\mathbb{R}^2$. For any two triangles T and S in the plane, define $T \sim S$ to mean that T is congruent to S in the usual geometric sense.

Which of the following are equivalence relations? If not an equivalence relation, which of the three axioms (reflexivity, symmetry, transitivity) hold?

Example 5. In the set $\mathbb{Z}^+$, let $x \sim y$ mean x divides y.

Example 6. $S = \mathbb{Z}$,

$$a \sim b \text{ means } (a, b) = 1.$$

Example 7. In the set $\mathbb{R}$, let $x \sim y$ mean $x - y$ is a rational.

Example 8. In the set $\mathbb{R}$, let $x \sim y$ mean $|x| = |y|$.

Example 9. In the set $\mathbb{R}$, let $x \sim y$ mean $x \leq y$.

Example 10. In the set $\mathbb{R}$, let $x \sim y$ mean $|x - y| \leq 1$.

Example 11. In the set X, let $x \sim y$ hold for all $x, y \in X$.

Example 12. In the set $\mathbb{Z} \times \mathbb{Z} - \{0\}$, let $(a, b) \sim (c, d)$ mean $ad = bc$.

Example 13. In the set $\mathbb{R}^2$, let $P \sim Q$ mean that points P and Q lie on the same line through the origin.

Example 14. In the set $\mathbb{R}^2$, let $P \sim Q$ mean that points P and Q lie on the same vertical line.

Example 15. In the set $\mathbb{R}^2$, let $P \sim Q$ mean that points P and Q lie on the same line.

Example 16. In the set $\mathbb{R}^2$, let $(a, b) \sim (c, d)$ mean $a^2 + b^2 = c^2 + d^2$.

Example 17. For the set of all triangles in $\mathbb{R}^2$, let $T \sim S$ mean that T is similar to S in the usual geometric sense.

Example 18. For the set of all subsets of $\mathbb{R}$, let $A \sim B$ mean that $A \cap B = \emptyset$.

Example 19. For the set of all subsets of $\mathbb{R}$, let $A \sim B$ mean that $A \subseteq B$.

Proposition. Suppose $\sim$ is a transitive relation on a set S. If $x_1 \sim x_2$, $x_2 \sim x_3$, $\cdots$, $x_{n-1} \sim x_n$, then $x_1 \sim x_n$.

Example 20. For the set of all people on earth, let $P_1 \sim P_2$ mean that P_1 looks like P_2.

Example 21. For the set of all people on earth, let $P_1 \sim P_2$ mean that P_1 has the same biological parents as P_2.

Example 22. For the set of all functions $f : \mathbb{R} \to \mathbb{R}$, let $f \sim g$ mean that f and g differ by a constant, that is, there is some constant c such that $f(x) = g(x) + c$ for all $x \in \mathbb{R}$.

Example 23. For the set of all functions $f : \mathbb{R} \to \mathbb{R}$, let $f \sim g$ mean that f is a constant multiple of g, that is, there is some constant k such that $f(x) = k \cdot g(x)$ for all $x \in \mathbb{R}$.

Example 24. For the set of all differentiable functions $f : \mathbb{R} \to \mathbb{R}$, let $f \sim g$ mean that $f' = g'$, where $'$ denotes the derivative.

Question 25. Here is a "proof" that symmetry and transitivity imply reflexivity:

Suppose $\sim$ is symmetric and reflexive on a set S. By symmetry we have $x \sim y$ implies $y \sim x$ and by transitivity, we can deduce from $x \sim y$ and $y \sim x$ that $x \sim x$, thereby establishing reflexivity.

What, if anything, is wrong with this proof?

Example 26. For the set of all subsets of $\mathbb{R}$, let $A \sim B$ mean that $A \overset{\subset}{\neq} B$ or $B \overset{\subset}{\neq} A$. Show that $\sim$ is symmetric and transitive. Is it relexive?

Example 27. Given a function $f : X \to Y$ and define $a \sim b$ to mean $f(a) = f(b)$.

Equivalence Classes

If $\sim$ is an equivalence relation on a set S, then define the **equivalence class** $[a]$ to be the set $\{b \in S : a \sim b\}$.

Theorem. Let $\sim$ be an equivalence relation on the set S. Then for any two equivalence classes $[a]$ and $[b]$,

$$\begin{cases} [a] = [b] & \text{if } a \sim b \\ [a] \cap [b] = \emptyset & \text{if } a \not\sim b \end{cases}.$$

In particular, if we define $S/\sim$, the **factor set** of the equivalence relation $\sim$, to be

$$S/\sim \overset{\text{def}}{=} \{[a] : a \in S\},$$

then the theorem above states that any two equivalence classes are either identical or disjoint. Consequently,S can be partitioned into disjoint equivalence classes in $S/\sim$.

For example, $\mathbb{Z}$ is the disjoint union of the equivalence classes $[0]_3$, $[1]_3$, and $[2]_3$.

Exercise. Describe in as simple way as possible the equivalence classes of the preceding examples which turned out to be equivalence relations.

Theorem (from book). If $f : S \to T$ is any function, let $\sim$ be the equivalence relation defined on S be letting $x_1 \sim x_2$ if $f(x_1) = f(x_2)$, for all $x_1, x_2 \in S$. Then there is a one–to–one correspondence between the elements of the image $f(S)$ of S under f and the equivalence classes in the factor set $S/\sim$.

Example. [The last name function.] Let $\mathcal{P}$ denote the set of people on earth and let $\mathcal{N}$ denote the set of all possible names. Let $L : \mathcal{P} \to \mathcal{N}$ be the function defined by $L(P) =$ the last name of person P. (We have to restrict people with one name, such as Cher, from our domain $\mathcal{P}$.) We will get lots of last names, such as Smith, Jones, and Pickleheimer. Two people are related by $\sim$ if they have the same last name. The above theorem states there is a one–to–one correspondence:

Name		Equivalence Class
Smith	$\leftrightarrow$	{people named Smith}
Jones	$\leftrightarrow$	{people named Jones}
Pickleheimer	$\leftrightarrow$	{people named Pickleheimer}

Example. [The remainder function mod 3.] Let $R : \mathbb{Z} \to \{0, 1, 2\}$ be the function defined by $R(n) =$ the remainder on dividing n by 3. The above theorem states there is a one–to–one correspondence:

Remainder		Equivalence Class
0	$\leftrightarrow$	$\{\ldots, -6, -3, 0, 3, 6, 9, \ldots\}$
1	$\leftrightarrow$	$\{\ldots, -5, -2, 1, 4, 7, 10, \ldots\}$
2	$\leftrightarrow$	$\{\ldots, -4, -1, 2, 5, 8, 11, \ldots\}$

That is, each remainder $r = 0, 1, 2$ is associated with the congruence classe $[r]_3$.