Math 32404: Advanced Calculus II
Problem set 2, due on Wednesday, October 4th, at 6pm.
Solutions turned in after 4:05pm are late and get half credit.

Your solutions should include explanations that would be understandable and convincing to your classmates. Ideally, your solutions should look like examples and proofs in our textbook.

Please cite any outside sources (books, webpages, experts) that you consult, any technology (calculators, computer software, slide rules) that you use, and any classmates that you collaborate with.

1. Let $s_0 := (1,2) \in \mathbb{R}^2$, and let $s_{n+1} := s_n + 2^{-n}(\cos(n), \sin(n))$. Does this sequence converge? Prove your answer!

2. For a sequence (s_n) in $\mathbb{R}^p$, let $d_n := \|s_n - s_{n+1}\|$. Prove that the sequence (s_n) converges if and only if the sequence (d_n) converges to 0; or find a counterexample.

3. Suppose that (s_n) and (t_n) are two sequences in $\mathbb{R}^2$, and define another sequence (u_n) by $u_{2m} := s_m$ and $u_{2m+1} := t_m$.
   (a) Prove that (u_n) converges whenever both (s_n) and (t_n) converge; or find a counterexample.
   (b) Prove that (u_n) converges to u whenever both (s_n) and (t_n) converge to u; or find a counterexample.

4. Let (f_n) be a sequence of functions from $\mathbb{R}^p$ to $\mathbb{R}^q$, and let $D_1 \subset D_2 \subset \ldots \subset \mathbb{R}^p$ be a countable collection of open subsets D_i of $\mathbb{R}^p$ with $D_i \subset D_{i+1}$ for all i; and let $D := \cup_i D_i$.
   (a) Suppose that the sequence (f_n) converges on each D_i. Prove that (f_n) converges on D; or fnd a counterexample.
   (b) Suppose that the sequence (f_n) converges uniformly on each D_i. Prove that (f_n) converges uniformly on D; or fnd a counterexample.

5. Let (f_n) be a convergent sequence of functions such that $f_n \in B_{pq}(D)$ for all n. Prove that the norm $\|f\|_D$ of the limit f of this sequence is equal to the limit of the sequence $s_n := \|f_n\|_D$ of norms of the functions in the sequence; or find a counterexample.

6. Suppose that (f_n) is a convergent sequence of constant functions on D. Prove that the limit function f is also constant; or find a counterexample.

7. Suppose that (f_n) is a convergent sequence of differentiable functions on $[1,2] \subset \mathbb{R}^1$. Prove that the limit function f is also differentiable; or find a counterexample.

8. Suppose that (f_n) is a convergent sequence of polynomials on D. Prove that the limit function f is also a polynomial; or find a counterexample.

For class etc

1. Prove the following statement or find a counterexample. If every sequemce in A has a convergent subsequence, then A is bounded.

2. Suppose that for every convergent sequence (s_n) in A, the limit of (s_n) is in A. Show that A must be closed, or find a counterexample.

3. Let (s_n) be a sequence in $\mathbb{R}^p$, and let $X := \{s_n : n \in \mathbb{N}\}$. Prove that the set of cluster points of X is equal to the set of limits of convergent subsequences of (s_n); or find a counterexample.

4. Suppose that (f_n) is a convergent sequence of bounded functions on D. Prove that the limit function f is also bounded; or find a counterexample.

5. Suppose that (f_n) is a convergent sequence of linear functions from $\mathbb{R}^2$ to $\mathbb{R}^3$, and let f be its limit. Let M_n be the 3-by-2 matrix representing f_n with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

   (a) Prove that the six entries of these matrices form six convergent sequences in $\mathbb{R}^1$; or find a counterexample.

   (b) Prove that the limit function f is linear; or find a counterexample.

   (c) If you proved part (b), what matrix represents f?

   (d) If you found a counterexample for part (b), is it also a counterexample to part (a)?

Chunk 1 leftovers

1. proof of cauchy-schwartz, geometric interpretation of addition and dot product in a plane,
2. any two vectors lie in a plane, how to find its equation: parametrized, level set, graph.
3. Which of the following are equivalent to "p is an interior point of A"?
   (a) A contains some open ball centered at p
   (b) A contains some closed ball centered at p
   (c) A contains some open cell centered at p
   (d) A contains some closed cell centered at p
   (e) A contains some open ball containing p
   (f) A contains some closed ball containing at p
   (g) A contains some open cell containing p
   (h) A contains some closed cell containing p
   (i) A contains some open set containing p
   (j) A contains some closed set containing p
   (k) v contains some neighborhood of p
4. Prove that every open set is a union of closed balls; or find counterexample.
5. Prove that every union of closed balls is an open set; or find counterexample.
6. Go through the proof of Theorem 10.6 (Bolzano-Weierstrass, infinite bounded set has a cluster point) with $B := \{(\frac{\sin(n)}{n+1}, \frac{n}{n+1}) : n \in \mathbb{N}\}$ bounded by the closed cell $[-1, 1] \times [0, 1]$.
7. Which of the 32 combinations of (not) open; (not) closed; (not) connected; (not) bounded; (not) convex; are possible? which of those are compact?
8. Let $A_n := \{v \in \mathbb{R}^n : 2 \leq ||v|| < 3\}$. For what n is A_n connected? For what n is A_n bounded?
9. Let $u \in \mathbb{R}^n$ be the vector all of whose coordinates are 1; let $B_n := \{v \in \mathbb{R}^n : 2 \leq v \cdot u < 3\}$. For what n is B_n connected? For what n is B_n bounded?
10. Prove that for any n and for any set $A \subset \mathbb{R}^n$, the complement of the boundary of A is disconnected; or find a counter example.
11. Find an uncountable closed set with no interior points; or prove that there aren't any.
12. Find an uncountable connected set with no interior points; or prove that there aren't any.
13. Find an uncountable open set with no interior points; or prove that there aren't any.
14. (should been a reading exercise) Find an open cover of $(0, 1)$ that has no finite subcover. Same for $\mathbb{Z}$ and $\mathbb{Q}$ and $\mathbb{Q} \cap [0, 1]$.