Math 32404: Advanced Calculus II
Problem set 4, due on Wednesday, November 8th, at 6pm.
Solutions turned in after 6:05pm are late and get half credit.

Your solutions should include explanations that would be understandable and convincing to your classmates. Ideally, your solutions should look like examples and proofs in our textbook.

Please cite any outside sources (books, webpages, experts) that you consult, any technology (calculators, computer software, slide rules) that you use, and any classmates that you collaborate with.

1. Let $f(x, y) := \|(x, y)\|$.
   (a) Compute the two partial derivatives of f; make sure to indicate their domain.
   (b) At what points (a, b) is f differentiable?
   (c) Find the equation of the plane P tangent to the graph of f above the point $(3, 4)$.
   (d) Find an everywhere differentiable parametrization $g_5 : \mathbb{R} \to \mathbb{R}^2$ of the level set
   $$S := \{(x, y) \in \mathbb{R}^2 : f(x, y) = 5\}$$
   of f at 5. That is, you should have $S = \{g_5(t) : t \in \mathbb{R}\}$.
   (e) Compute the two partial derivatives of g_5.
   (f) Find the equation of the line L in $\mathbb{R}^2$ tangent to S at the point $(3, 4)$.
   (g) Verify that $\{(x, y, 5) : (x, y) \in L\} \subset P$.
   (h) Fix a real number $r > 0$; find an everywhere differentiable parametrization $g_r : \mathbb{R} \to \mathbb{R}^2$ of the level set of f at r.
   (i) Find an everywhere differentiable parametrization $g : \mathbb{R}^2 \to \mathbb{R}^3$ of the graph of f.
   (j) Find an everywhere differentiable function $h : \mathbb{R}^2 \to \mathbb{R}$ with the same level sets as f; that is, such that a subset $A \subset \mathbb{R}^2$ is a level set of h if and only if it is a level set of f.

2. Let $f(x, y, z) := (xyz^2 - 4y^2, 3xy^2 - yz)$.
   (a) Compute the Jacobian matrix of f at $(1, 2, 3)$.
   (b) Let $g : \mathbb{R}^3 \to \mathbb{R}^5$ be given by $g(x, y, z) := (x, y, z, f(x, y, z))$.
   Compute the Jacobian matrix of g at $(1, 2, 3)$.
   (c) Let $h : \mathbb{R}^5 \to \mathbb{R}^3$ be given by $h(x, y, z, u, v) := f(x, y, z) - (u, v)$.
   Compute the Jacobian matrix of h at $(1, 2, 3, 2, 6)$.
   (d) Find equations describing the space tangent to the graph of f at the point $(1, 2, 3, f(1, 2, 3))$.
   (e) Find equations describing the space tangent to the image of g at the point $g(1, 2, 3)$.
   (f) Find equations describing the space tangent to the level set $h(x, y, z, u, v) = 0$ at the point $(1, 2, 3, 2, 6)$.

3. Let f be the function
   $$f(x, y) := \begin{cases} \frac{x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$
   (a) Show that all directional derivatives of f at $(0, 0)$ exist.
   (b) Compute the two partial derivatives of f.
   (c) Does this function satisfy the conclusion of Corollary 39.7 at $(0, 0)$?
   (d) Which hypothesis of Theorem 39.9 (if any) is not satisfied by this function at $(0, 0)$?
   (e) What step in the proof of Theorem 39.9 (if any) does not work for this function?
   (f) Answer the same questions (a) - (e) for the function defined in exercise 39.H.

4. Consider the function $E : \mathbb{R}^9 \to \mathbb{R}$ that returns the determinant of a 3-by-3 input matrix; and another function $I : \mathbb{R}^9 \to \mathbb{R}^9$ that returns the inverse of a 3-by-3 input matrix.

   (a) What is the domain of E? At what points of its domain is E differentiable?

   (b) What is the domain of I? Show that it is open. At what points of its doamin is I differentiable?

   (c) Compute the Jacobian matrix and the Jacobian determinant of I at the identity matrix.

5. Fix an everywhere differentiable $f : \mathbb{R}^3 \to \mathbb{R}$ and a point (a,b,c). What extra hypothesis is needed to show that the gradient of f at (a, b, c) is perpendicular to the plane tangent to the level set $f(x, y, z) = f(a, b, c)$ at the point (a, b, c)?