Math 32404: Advanced Calculus II
Reading exercises 3, due on Wednesday, October 11th.

Read sections 20-24 in our textbook, including the exercises in the textbook, and solve the exercises below as you go along. Your solutions will not be collected, but a very short in-class quiz on the due date will contain one of these exercises, or one very very similar to it.

1. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $f(x, y) := (g(x, y), h(x, y))$ for some functions $g, h : \mathbb{R}^2 \to \mathbb{R}$. Show that f is continuous at $(0, 0)$ if and only if both h and g are continuous at $(0, 0)$.
2. Let f be the Dirichlet function, and let $g(x) := \sin(x)f(x)$. At what $a \in \mathbb{R}$ is g continuous?
3. Solve exercise 20Q in our textbook.
4. Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) := \frac{xy}{x^2+y^2}$ when $(x, y) \neq (0, 0)$, and let $f(0, 0) := 0$.
   (a) Compute the following (single-variable!) limits: $\lim_{t\to 0} f(t, 0)$, $\lim_{t\to 0} f(0, t)$, $\lim_{t\to 0} f(t, t)$.
   (b) Is this function continuous at $(0, 0)$?
   (c) Sketch a graph of this function; remember to cite technology if you use it.
5. Find and correct the error in Theorem 20.6 on p. 143
6. Check the definition of *linear function* for the following functions: $f(x, y) = 2x+3y+17$, $g(x, y, z) = (y, x)$, $h(x, y, z) := (|y|, |z|)$.
7. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be the function that returns the sum of its three inputs and the avegrage of its three inputs. Show that f is linear and find the matrix representing it. Find some A that satisfies the conclusion of Theorem 21.3 for this function.
8. From the sections you are reading this week, collect all properties of functions that are equivalent to continuity. Take your favourite discontinuous function and verify that each of the properties in your collection fails.
9. Let $d : \mathbb{R}^4 \to \mathbb{R}$ be the determinant function for 2×2 matrices; that is, let $d(x, y, z, w) := xw - yz$. Show that the set of invertible matrices (that is, $\{M \in \mathbb{R}^4 : d(M) \neq 0\}$) is disconnected.
10. Find a continuous function $f : \mathbb{R}^{17} \to \mathbb{R}^{16}$ and a compact set $C \subset \mathbb{R}^{16}$ such that $f^{-1}(C)$ is not compact.
11. Find a continuous function $f : \mathbb{R}^{17} \to \mathbb{R}^{16}$ and an open set $D \subset \mathbb{R}^{17}$ such that $f(D)$ is not open.
12. Which of the following functions is uniformly continuous on $\mathbb{R}^2$?
    $g(x, y) := \sin(x + y)$, $h(x, y) := x + y$, $w(x, y) := x^2 + y^3$, the function f from exercise 4 above.
13. Which of the following functions is uniformly continuous on the closed cell $[2, 3] \times [5, 6]$?
    $g(x, y) := \sin(x + y)$, $h(x, y) := x + y$, $w(x, y) := x^2 + y^3$, the function f from exercise 4 above.
14. Show that the function $f(x) := \sqrt{x}$ is not Lipschitz on the interval $[0, 1]$. is it uniformly continuous on $[0, 1]$?
15. Consider the function $f(x, y) := (\frac{y-1}{2}, \frac{-x-3}{2})$.
    (a) Find the unique fixed point (a, b) of f.
    (b) Let $a_1 := (1, 1)$, and let $a_{n+1} := f(a_n)$ for each $n \geq 1$. Compute a_n for $n \leq 10$. Compare to part (a).
    (c) Show that f is a contraction.
    (d) Sketch the points a_n obtained in part (b); describe f geometrically; compute a_{99}.