Math 32404: Advanced Calculus II Problem set 0, due on Thursday, February 1st, at 4pm. Solutions turned in after 4:05pm are late and get half credit.

Your solutions should include explanations that would be understandable and convincing to your classmates. Ideally, your solutions should look like examples and proofs in our textbook.

Please cite any outside sources (books, webpages, experts) that you consult, any technology (calculators, computer software, slide rules) that you use, and any classmates that you collaborate with.

Short questions.

- 1. Sketch the following sets on the xy-plane; no two are the same.
 - (a) $\{(x, y) : \forall n \in \mathbb{N} \text{ if } x < n \text{ then } n < y\}$
 - (b) $\{(x, y) : \forall n \in \mathbb{N} \text{ if } n < y \text{ then } x < n\}$
 - (c) $\{(x, y) : \forall n \in \mathbb{N} \text{ if } x < n \text{ then } y < n\}$
 - (d) $\{(x, y) : \forall n \in \mathbb{N} \text{ if } y < n \text{ then } x < n\}$
- 2. The statement "Every real number is smaller than some rational number" is ambiguous. One of the following two interpretations is true, and the other is false. Prove the true one; negate the false one and prove the negation.
 - (i) $\forall r \in \mathbb{R} \exists q \in \mathbb{Q} r < q$
 - (ii) $\exists q \in \mathbb{Q} \, \forall r \in \mathbb{R} \, r < q$
- 3. Find the 2-by-3 matrix M such that

$$M \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} d \\ e \end{bmatrix}$$

whenever $(ax^2 + bx + c)' = dx + e$.

- 4. (a) Parametrize the curve defined by the equation $4x^2 + 9y^2 = 25$.
 - (b) Find the equation of the curve parametrized by $h(t) := (t^2, t^3)$.
 - (c) Find a function whose graph is one of these two curves.

Long questions on the back.

5. Read the first seven sections of our textbook, most of which should be review. Make a list of definitions and theorems in these section that are new to you.

6.

Let
$$u_n(x) := \begin{cases} x^n \sin(\frac{1}{x}) & \text{if } x \neq 0\\ 0 & \text{if } x = 0 \end{cases}$$

Recall that for a function F and real numbers a and L, the notation " $\lim_{x\to a} F(x) = L$ " means:

"For every real number $\epsilon \ge 0$ there exists a real number $\delta \ge 0$ such that for all real numbers x, if $0 \le |x-a| \le \delta$, then $|F(x) - L| \le \epsilon$."

Evaluate the limits in parts (a), (b), and (c).

If the limit exists, find a δ satisfying the above definition for $\epsilon = 0.1$.

If the limit does not exist, find an ϵ for which no δ satisfies the above definition.

- (a) For n = 0, 1, 2, ..., consider $\lim_{x \to 0} u_n(x)$.
- (b) For n = 0, 1, 2, ..., consider $\lim_{x \to 0} \frac{u_n(x)}{x}$.
- (c) For n = 0, 1, 2, ..., consider $\lim_{x \to 0} u'_n(x)$.
- (d) For what non-negative integers n is the function u_n continuous at 0?
- (e) For what non-negative integers n is the function u_n differentiable at 0?
- (f) For what non-negative integers n is the derivative of the function u_n continuous at 0?
- 7. Let a be a real number, and let $g(x) := \frac{x}{x^2+a}$. Answer the following questions in terms of a.
 - (a) What is the domain of g?
 - (b) Plot five points on the graph of g.
 - (c) Compute the limits of g at $\pm \infty$.
 - (d) For each real number r, if g(r) is not defined, compute the two one-sided limits of g(x) as x tends to r.
 - (e) Compute g'(x).
 - (f) Find and classify the critical points of g.
 - (g) How many inflection points does the graph of g have?
 - (h) Sketch the graph of g.

8. Let
$$f(x,y) := \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } x, y) = (0,0) \end{cases}$$

- (a) For what real numbers b is the level set $\{(x, y) : f(x, y) = b\}$ empty?
- (b) For what real number z does the level set $\{(x, y) : f(x, y) = z\}$ consist of two lines?
- (c) All other level sets of the this function are conic sections of the same kind; which kind?
- (d) Sketch five level sets of this function.
- (e) What is the relation between this function g and the function f from the previous problem?
- (f) Sketch the graph of g; remember to cite technology if you use it.
- (g) Describe the graph of g in a few sentences.
- (h) Convert this story into polar coordinates; does this help you answer parts (f) and (g)?
- (i) Suppose that x and y are horizontal directions, and the value of f is the height. If you put a ball on the graph of f at the point (0, 2, 0), which way will it roll?