Math 32404: Advanced Calculus II
Problem set 3, due on Thursday, March 8,, at 4pm.
Solutions turned in after 4:05pm are late and get half credit.

Your solutions should include explanations that would be understandable and convincing to your classmates. Ideally, your solutions should look like examples and proofs in our textbook.

Please cite any outside sources (books, webpages, experts) that you consult, any technology (calculators, computer software, slide rules) that you use, and any classmates that you collaborate with.

1. (a) Consider the preimage of the image of a set S under a function f. Show that $f^{-1}(f(S)) = S$, or find a counterexample.
   (b) Consider the image of the preimage of a set S under a function f. Show that $f(f^{-1}(S)) = S$, or find a counterexample.
2. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a continuous function. Fill in each of two blanks in the following statement with one of the words provided to make the statement true; and prove the statement.

   "If $A \subset \mathbb{R}^3$ is [ ] , then $f^{-1}(f(A))$ is [ ] ."

   Words: open, closed, bounded, connected, compact.
   Find counterexamples to the other 24 ways of filling in the blanks.
3. Suppose the $S \subset \mathbb{R}^2$ is a connected set containing the points $(17, -17)$ and $(-17, 17)$. Use the Intermediate Value Theorem (you'll need to invent a suitable function) to prove that S must contain a point on the line $x = y$; or find a counterexample.
4. Prove one of the following and find a counterexample to the other.
   (a) Fix a function f with domain D which is continuous on D, and a sequence (a_n) in D that converges to a point a in D. Then the sequence (b_n) where $b_n := f(a_n)$ converges to $f(a)$.
   (b) Suppose that a sequence (f_n) of continuous functions from $\mathbb{R}^p$ to $\mathbb{R}^q$ converges to a function f on all of $\mathbb{R}^p$. Then for any $a \in \mathbb{R}^p$, the sequence (b_n) where $b_n := f_n(a)$ converges to $f(a)$.
5. Investigate the relation between between m in Corollary 22.8 and the norm defined in exercises 21.L and 21.M with some simple examples.
6. Find a function of two variables that is continuous in each variable alone at all points, but discontinuous as a function of two variables, also at all points; or prove that this is impossible.