Math A4400: Mathematical Logic
5th problem set, due at 2pm on wednesday, october 16th.

Bring your solutions class, or slide them under the door of my office NAC 6278.

1. Let S be the signature with two binary function symbols P and M. Let $\mathcal{A}$ be the S-structure with universe $A := \mathbb{R}$, with $M^{\mathcal{A}}(a,b) := a \cdot b$, and $P^{\mathcal{A}}(a,b) := a + b$. Let $\mathcal{B}$ be the S-substructure of $\mathcal{A}$ with universe $B := \mathbb{Q}$. Describe all S-homomorphisms between these two structures, in both directions, and prove that your list includes all of them.

2. Let S and $\mathcal{A}$ be as in problem 1. Let α be the $\mathcal{A}$-assignment given by $\alpha(v_n) := 2^n$. Describe $\{t^{\mathcal{A}}[\alpha] \mid t \text{ is an } S\text{-term}\}$. As usual, prove your answer.

3. Let S be a signature, and let $\mathcal{A}$ be an S-structure. Let E be an equivalence relation on the universe A of $\mathcal{A}$, and let $\pi : A \to A/E$ be the usual surjection from A to the set of E-equivalence classes.

   (a) Find the necessary and sufficient conditions for there to be an S-structure $\mathcal{B}$ with universe A/E such that $\pi : \mathcal{A} \to \mathcal{B}$ is a positive S-homomorphism (see Remark 2.3.27 in our book; this is what "homomorphism" was defined to mean in lecture).

   (b) Find the necessary and sufficient conditions for π above to be an S-homomorphism in the sense of Definition 2.3.26 in our book.

   (c) Let S be the signature with one binary function symbol M and one binary relation symbol D. Let $\mathcal{A}$ be the S-structure with universe $A := \mathbb{Q} \setminus \{0\}$, with $M^{\mathcal{A}}(a,b) := a \cdot b$, and with $D^{\mathcal{A}} := \{(a,b) \mid a \text{ divides } b\}$. Let $E := \{(a,b) \mid b = c^2 a \text{ for some } c \in \mathbb{Q}\}$. Describe the quotient structure $\mathcal{B}$. Is π a homomorphism?

4. Ask an interesting question about this week's material and try to answer it. This question is as serious as the rest of them!

bonus Let S be a signature with one unary function symbol f, and let S' be a signature with one binary relation symbol R. For any S-structure $\mathcal{A}$, define the *associated S'-structure* $\mathcal{A}'$ by $A' := A$ and

$$R^{\mathcal{A}'} := \{(a,b) \in A^2 \mid f^{\mathcal{A}}(a) = b\}$$

Is every S'-structure associated to some S-structure? If $g : A \to B$ is an S-homomorphism between S-structures $\mathcal{A}$ and $\mathcal{B}$, does it follow that g is also an S'-homomorphism between the associated S'-structures? Is the converse of the last statement true?