Math A4400: Mathematical Logic
6th problem set, due at 2pm on wednesday, october 23nd.
Bring your solutions class, or slide them under the door of my office NAC 6278.

1. Suppose that X, Y, and Z are symbols in a signature S.

   (a) Could both XYZ and XZY be S-formulas? If so, what kinds of symbols would X, Y and Z have to be?

   (b) Could both XYZ and ZYX be S-terms? If so, what kinds of symbols would X, Y and Z have to be?

   List all possibilities and carefully prove your answers.

2. Let S be a signature with one binary function symbol F and no other symbols. Let $\phi := \forall x\, Fxy \doteq x$ and $\psi := \forall y\, Fxy \doteq x$ be two S-formulas. Consider the following three S-structures with universe $\mathbb{Z}$:

   - $F^{\mathcal{T}}(a, b) := a \cdot b$;
   - $F^{\mathcal{P}}(a, b) := a + b$;
   - $F^{\mathcal{F}}(a, b) := a$.

   For each of the six possible combinations of formula and structure, describe the set of assignments that satisfy that formula in that structure.

3. Let S be a signature, x a variable, and τ an S-term. Prove or refute the following.

   (a) If t is another S-term, and t' is the expression obtained by replacing every instance of x in t with τ, then t' is also an S-term.

   (b) If α is an S-formula, and β is the expression obtained by replacing every instance of x in α with τ, then β is also an S-formula.

4. Ask an interesting question about this week's material and try to answer it. This question is as serious as the rest of them!

Bonus Let S be the signature with two binary function symbols $\dot{+}$ and $\dot{\times}$ and two constant symbols $\dot{0}$ and $\dot{1}$. Let $\mathcal{N}$ be the S-structure with universe $\mathbb{N}$ and the usual interpretations of the symbols of S.

   (a) Show that for every natural number n, there is an S-term $\dot{n}$ with no variables such that $\dot{n}^{\mathcal{N}}[\alpha] = n$ for any $\mathcal{N}$-assignment α.

   (b) Suppose that $\mathcal{A}$ is another S-structure, and suppose that for any atomic S-formula ϕ with no variables, any $\mathcal{A}$-assignment α, and any $\mathcal{N}$-assignment ν, $\models_{\mathcal{A}} \phi[\alpha]$ if and only if $\models_{\mathcal{N}} \phi[\nu]$. Show that there exists a unique S-embedding from $\mathcal{N}$ to $\mathcal{A}$.