Math A4400: Mathematical Logic

7th problem set, due at 2pm on wednesday, november 20th.

Bring your solutions class, or slide them under the door of my office NAC 6278.

1. Let S be a signature with one ternary relation symbol R and no other symbols. For each of the following four S-structures $\mathcal{A}_i$, find an S-sentence θ_i such that $\models_{\mathcal{A}_i} \theta_i$ and $\not\models_{\mathcal{A}_j} \theta_i$ for any $j \neq i$.
   (a) $A_1 := \mathbb{Q}$ and $R^{\mathcal{A}_1} := \{(a,b,c) \in \mathbb{Q}^3 \mid a+b=c\}$.
   (b) $A_2 := \mathbb{Q}$ and $R^{\mathcal{A}_2} := \{(a,b,c) \in \mathbb{Q}^3 \mid a \cdot b = c\}$.
   (c) $A_3 := \mathbb{Q}$ and $R^{\mathcal{A}_3} := \{(a,b,c) \in \mathbb{Q}^3 \mid a \leq b \leq c\}$.
   (d) $A_4 := \mathbb{Z}$ and $R^{\mathcal{A}_4} := \{(a,b,c) \in \mathbb{Z}^3 \mid a \leq b \leq c\}$.

2. Let S be the signature with one binary relation Q, and let $\mathcal{A}$ be the S-structure with universe $\{a \in \mathbb{Z} \mid a \geq 17\}$ and $Q^{\mathcal{A}} := \{(a,b) \in \mathbb{Z}^2 \mid a < b\}$. Let θ be the S-formula $Qv_1v_2 \wedge \neg\exists v_3\,(Qv_1v_3 \wedge Qv_3v_2)$. For each of the formulas ϕ_i below, find an $\mathcal{A}$-assignment α_i such that $\models_{\mathcal{A}} \phi_i[\alpha_i]$ and $\not\models_{\mathcal{A}} \phi_j[\alpha_i]$ for any $j \neq i$.
   (a) $\phi_1 := \forall v_5\,(Qv_5v_7 \rightarrow \dot{=}\, v_5v_7)$.
   (b) $\phi_2 := \exists v_9\,((\phi_1)_{v_7}(v_9) \wedge (\theta_{v_1}(v_9))_{v_2}(v_7))$
   (c) $\phi_3 := \neg\forall v_4\,(Qv_4v_7 \rightarrow ((\phi_1)_{v_7}(v_4) \vee (\phi_2)_{v_7}(v_4)))$
   (d) $\phi_4 := \exists v_5\,\exists v_6\,(Qv_5v_6 \wedge Qv_6v_7 \wedge \forall v_8\,(Qv_8v_7 \rightarrow (\dot{=}\, v_8v_5 \vee \dot{=}\, v_8v_6)))$

3. Let S be the signature with one binary function symbol $\times$, and let $\mathcal{A}$ be the S-structure with universe $\mathbb{Q}$ and with $\times^{\mathcal{A}}(a,b) := a \cdot b$. Show that there is no S-formula ϕ such that $\models_{\mathcal{A}} \phi[\alpha]$ if and only if $\alpha(v_1) + \alpha(v_2) = \alpha(v_3)$. Hint: consider automorphisms of $\mathcal{A}$.

4. (The last problem from the second midterm.) Let S be a signature with a binary function symbol F. Let $\mathcal{A}$ be the S-structure with universe $A := \mathbb{Z}$ and $F^{\mathcal{A}}(a,b) := a+b$. Let $B := \{-1, 1\}$, and let $h : A \rightarrow B$ be given by
$$h(n) := \begin{cases} 1 & \text{if } n \geq 0 \\ -1 & \text{if } n < 0 \end{cases}.$$
   If there is an S-structure $\mathcal{B}$ with universe B for which h is an S-homomorphism, determine $F^{\mathcal{B}}$. Otherwise, prove that there is no such S-structure

5. Ask an interesting question about the material we covered since the midterm and try to answer it. This question is as serious as the rest of them!