Math A4400: Mathematical Logic

8th and last problem set, due at 2pm on wednesday, december 11th.

Bring your solutions class, or slide them under the door of my office NAC 6278.

1. Let $S := \{+, \times, 0, 1\}$ be the signature of rings, let $\mathcal{N}$ be the S-structure with universe $\mathbb{N}$ and usual interpretations of the symbols in S, and let t_n be S-terms with no variables such that $t_n^{\mathcal{N}} = n$. Let $\mathcal{A}$ be an S-structure and α an $\mathcal{A}$-assignment satisfying $Th(\mathcal{N}) \cup \{x \neq t_n \mid n \in \mathbb{N}\}$.
   (a) Show that such a structure $\mathcal{A}$ exists, using the Compactness of First-Order Logic; as we did in class.
   (b) Define $h : \mathbb{N} \to A$ by $h(n) := t_n^{\mathcal{A}}$. Show that h is an S-embedding. These elements $t_n^{\mathcal{A}}$ are the *standard elements of* $\mathcal{A}$; others are *non-standard.*
   (c) Find an S-formula $\phi_{\leq}$ such that for any $\mathcal{N}$-assignment β, $\models_{\mathcal{N}} \phi_{\leq}[\beta]$ if and only if $\beta(v_1) \leq \beta(v_2)$. Show that $\phi_{\leq}$ also defines an ordering on the universe A of $\mathcal{A}$.
   (d) Are any nonstandard elements of $\mathcal{A}$ less that $0^{\mathcal{A}}$ in the sense of the ordering given by $\phi_{\leq}$? Between $t_3^{\mathcal{A}}$ and $t_4^{\mathcal{A}}$?
   (e) Find an S-formula η such that for any $\mathcal{N}$-assignment β, $\models_{\mathcal{N}} \eta[\beta]$ if and only if $\beta(v_1)$ is an even number. Show that there is an $\mathcal{A}$-assignment α such that $\models_{\mathcal{A}} \eta[\alpha]$ and $\alpha(v_1)$ is non-standard. For what other properties of natural numbers (besides "even") can you do the same?

2. Use the Compactness theorem for first-order logic to show that the class of connected graphs is not axiomatizable. That is, there is no signature S and no set of S-sentences T such that every model of T is a connected graph and every connected graph is a model of T.

3. Let $S := \{+, \times, 0, 1\}$ be the signature of rings.
   (a) Write down a set T of S-sentences whose models are algebraically closed fields. (That is, all models of T are algebraically closed fields, and all algebraically closed fields are models of T.)
   (b) Use the Compactness theorem for first-order logic to show that there is no set T' of S-sentences such that its models are non-algebraically closed fields.

4. Let $S := \emptyset$ be the the first-order signature with no non-logical symbols.
   (a) Give an explicit enumeration $\{\theta_n \mid n \in \mathbb{N}\}$ of all S-sentences. In your enumeration, what is θ_0? θ_1? θ_{17}?
   (b) What S-structure will be produced by the Henkin construction starting with $\Gamma := \emptyset$ with your enumeration?

5. Ask an interesting question about first-order logic and try to answer it. This question is as serious as the rest of them!