Math A4400: Mathematical Logic
Problem set 0, due at 2pm on thursday, february 11th.
Solutions turned in after 2:05pm are late and get half credit.

You may also bring your solutions in my office NAC 6278; if I am not there, slide them under the door.

Questions labeled with * are somewhat harder.

All page numbers and chapter numbers refer to Mathematical Logic Lecture Notes by van den Dries. This problem set is about section 2.1.

1. Prove that the number of propositional atoms in a proposition is always one more than the number of $\wedge$s and $\vee$s in that proposition; or find a counterexample.

2. Let Max_k and min_k and be the largest and smallest numbers of propositional atoms in a proposition of length k. For example, $Max_1 = min_1 = 1 = Max_2 = min_2 = min_3 \neq Max_3 = 2$.
   (a) What are Max_k and min_k for $k = 3, 4, 6$?
   (b) What are Max_{17} and min_{17}? Prove your answer!
   (c) Find and prove formulas for these two functions; use them to evaluate Max_{2016} and min_{2016}.

* Prove that for any proposition $a_1 a_2 a_3 \dots a_n$ and any $m \lneq n$, the word $a_1 a_2 a_3 \dots a_m$ is not a proposition.

3. Use (*) to prove that for any proposition $a_1 a_2 a_3 \dots a_n$ and any $i \leq n$, there is a unique j such that $i \leq j \leq n$ and $a_i a_{i+1} \dots a_j$ is a proposition.

4. Let $bits : \text{Prop}(A) \to \mathcal{P}(\text{Prop}(A))$ be the function defined recursively as follows.

base of recursion: For a propositional atom a, $bits(a) := \emptyset$;

recursive steps: For any two propositions α and β,

$$bits(\neg\alpha) := bits(\alpha) \cup \{\alpha\} \quad \text{and} \quad bits(\wedge\alpha\beta) = bits(\vee\alpha\beta) := bits(\alpha) \cup \{\alpha\} \cup bits(\beta) \cup \{\beta\}.$$

Show that the proposition $a_i a_{i+1} \dots a_j$ in problem 3 is always an element of $bits(a_1 a_2 a_3 \dots a_n)$; or find a counterexample. Explain in words what the function $bits$ returns.