Math A4400: Mathematical Logic
Final Project, relying on Section 2.6.

Read Section 2.6 and these two pages, and think about each of the questions 1-15 for 5-10 minutes. Classify questions 1-15 as too boring, too hard, or just right for you; with at least 5 “just right.” Solve as many of the not-too-boring, not-too-hard parts as you can by Tuesday, May 10th. **Come to class on Tuesday May 10th,** or send your preferences (boring/hard/just right) to someone who will: we will assign parts to people. Solve all parts assigned to you by Tuesday May 17th.

1 introduction

The goal is to prove the Compactness Theorem of First Order Logic:

Theorem 1.1. *Every finitely satisfiable set of first-order sentences is satisfiable.*

Definition 1.1. *A set S of L-sentences is called* finitely satisfiable *if for any finite subset $S_0 \subset S$ there is an L-structure satisfying all formulae in S_0.*

We start with a first-order language L_0 and a finitely satisfiable set S_0 of L_0-sentences; we grow the language to L_ω by adding many new constant symbols, and grow the set S_0 to Σ with two extra properties: still finitely satisfiable, Σ is also *complete*, and *has constant witnesses.* We then build an L_ω-structre $\mathcal{U}$ out of the constants of L_ω, and show that it satisfies all sentences in Σ. Then the reduct of $\mathcal{U}$ to L_0 satisfies all sentences in S, and we are done.

2 the project

1. Given a first-order language L and a finitely satisfiable set T of L-sentences, let L' be a new language with lots of extra constant symbols, and T' be a set of L'-sentences as follows:

$$L' := L \cup \{C_\phi \mid \phi(x) \text{ is an } L\text{-formula } \}$$

$$T' := T \cup \{(\exists x\, \phi) \to \phi(C_\phi) \mid \phi(x) \text{ is an } L\text{-formula } \}$$

   Show that T' is finitely satisfiable.

2. Given a first-order language L_0 and a finitely satisfiable set S_0 of L_0-sentences, use the definitions above to define L_n and S_n inductively as follows:

$$L_{n+1} := L'_n \text{ and } S_{n+1} := S'_n$$

   Let

$$L_\omega := \cup_{n\in\mathbb{N}} L_n \text{ and } S_\omega := \cup_{n\in\mathbb{N}} S_n$$

   (a) Show that for all n, S_n is finitely satisfiable.
   (b) Show that S_ω is finitely satisfiable.
   (c) Show that for any L_ω-formula $\phi(x)$, there is a constant symbol C_ϕ in L_ω such that the sentence $(\exists x\, \phi) \to \phi(C_\phi)$ is in S_ω.

   Definition 2.1. *A set T of L-sentences is said to* have constant witnesses *if for every L-formula ϕ there is a constant symbol C_ϕ in L such that the sentence $(\exists x\, \phi) \to \phi(C_\phi)$ is in T.*

3. (a) Suppose that you have countably many symbols; show that there are only countable many finite strings of these symbols. Conclude that if a first-order language L has countably many non-logical symbols, then there are countably many L-formulae.

(b) Show that if there are countably many symbols in L, and countably many formulae in L, then there are countably many formulae in L' defined above.

(c) Given countable first-order languages L_n for $n \in \mathbb{N}$ such that $L_i \subset L_j$ for all $i \leq j$, show that $\cup_{n\in\mathbb{N}} L_n$ is countable. Conlude that L_ω defined above has countably many formulae.

4. Show that if a set T of L-sentences is finitely satisfiable, and θ is an L-sentence, then either $T \cup \{\theta\}$ or $T \cup \{\neg\theta\}$ is fintely satisfiable.

5. Show that if a set S_ω of L_ω-sentences is finitely satisfiable, then there exists a finitely satisfiable set Σ of L_ω-sentences such that $S_\omega \subset \Sigma$ and for every L_ω-sentence θ, either $\theta \in \Sigma$ or $\neg\theta \in \Sigma$.

Definition 2.2. *A set T of L-sentences is called* complete *is for every L-sentence θ, either $\theta \in T$ or $\neg\theta \in T$.*

6. So, we now have a set of L_ω-sentences $\Sigma \supset S$, which is complete, finitely satisfiable, and has constant witnesses.

Definition 2.3. *An L-sentence ψ is a* semantic consequence *of a set Σ of L-sentences if every L-structure $\mathcal{A}$ that satisfied all sentences in Σ also satisfies ψ.*

7. Suppose that a set T of L-sentences is complete and finitely satisfiable, and that an L-sentence ψ is a semantic consequence of a finite subset of T; show that $\psi \in T$.

8. Suppose that a set T of L-sentences is complete and finitely satisfiable; define two constant symbols C and D to be T-*equivalent* if the sentence $C = D$ is in T. Show that this is an equivalence relation.

9. Suppose that a set T of L-sentences is complete and finitely satisfiable; show that if C_i is T-equivalent to D_i for all $i \leq n$, and R is an n-ary relation symbol in L, then the sentence $R(C_1, \dots C_n)$ is in T if and only if the sentence $R(D_1, \dots D_n)$ is in T.

10. Suppose that a set T of L-sentences is complete and finitely satisfiable, and has constant witnesses. Show that for any constant symbols C_i in L and any n-ary function symbol f in L, there is a constant symbol C in L such that the sentence $f(C_1, \dots C_n) = C$ is in T.

11. Suppose that a set T of L-sentences is complete and finitely satisfiable; C_i is T-equivalent to D_i for all $i \leq n$; f is an n-ary function symbol in L; and sentences $f(C_1, \dots C_n) = C$ and $f(D_1, \dots D_n) = D$ are in T; show that C is T-equivalent to D.

12. We now construct an L_ω-structure $\mathcal{U}$ as follows: the universe U will consist of Σ-equivalence classes u_C of L_ω constant symbols C; we interpret L_ω as follows:

    - for a constant symbol C, we let $C^{\mathcal{U}} := u_C$
    - for an n-ary relation symbol R,

    $$R^{\mathcal{U}} := \{(u_{C_1}, u_{C_2}, \dots, u_{C_n}) \mid R(C_1, C_2, \dots C_n) \in \Sigma\}$$

    - for an n-ary function symbol f and an n-tuple $(u_{C_1}, u_{C_2}, \dots, u_{C_n})$ of elements in U, we let $f^{\mathcal{U}}(u_{C_1}, u_{C_2}, \dots, u_{C_n}) := u_C$ for some constant symbol C in L_ω such that $f(C_1, C_2, \dots C_n) = C$ is in Σ.

    Verify that the functions $f^{\mathcal{U}}$ are well-defined.

13. Suppose that a set T of L-sentences is complete and finitely satisfiable, and that the L-sentence $\forall x \phi(x)$ is in T, and that C is a constant symbol in L. Show that the L-sentence $\phi(C)$ is in T.

14. Show that $\mathcal{U}$ satisfies Σ. Hint: induct on the complexity of a *sentence* θ to show that $\mathcal{U} \models \theta$ if and only if $\theta \in \Sigma$. Further hint: use the fact that Σ has constant witnesses to deal with the quantifier indcution step.

15. Yay, we are done!