**Math A4400: Mathematical Logic**
**Problem set 2, due at 2pm on thursday, february 18th.**
**Solutions turned in after 2:05pm are late and get half credit.**

You may also bring your solutions in my office NAC 6278; if I am not there, slide them under the door.

Questions labeled with * are somewhat harder.

**All page numbers and chapter numbers refer to Mathematical Logic Lecture Notes by van den Dries. This problem set is about section 2.1.**

0. Read and thoroughly digest section 2.1, including exercises. If you are not thoroughly comfortable with equivalence relations, this is a good time to review them. Some parts of Lemmas 2.1.2 and 2.1.3 will definitely appear on the first midterm exam.

1. For this problem, let $A := \{p_0, p_1, p_2\}$.

   (a) Is there a proposition $\alpha \in \text{Prop}(A)$ such that for any truth assignment $V$,
   $V(\alpha) = T$ if and only if $V(p_i) = T$ for $i = 0, 1, 2$?

   (b) Is there a proposition $\beta \in \text{Prop}(A)$ such that for any truth assignment $V$,
   $V(\beta) = T$ if and only if $V(p_0)$, $V(p_1)$, and $V(p_2)$ are either all $T$ or all $F$?

   (c) Is there a proposition $\gamma \in \text{Prop}(A)$ with the following truth table?

   | $p_0$ | $p_1$ | $p_2$ | $\gamma$ |
   |---|---|---|---|
   | T | T | T | T |
   | T | T | F | F |
   | T | F | T | F |
   | T | F | F | T |
   | F | T | T | F |
   | F | T | F | F |
   | F | F | T | T |
   | F | F | F | F |

   (d) Prove that for any truth table on $A$, there is a proposition $\sigma \in \text{Prop}(A)$ with that truth table; or find a counterexample. That is, show that for any function

   $$w : \{ \text{ atomic truth assignments on } A \ \} \to \{T, F\},$$

   there is some $\sigma \in \text{Prop}(A)$ such that

   $$\bar{t}(\sigma) = w(t)$$

   for every atomic truth assignment $t$ on $A$; or show that there is no such $\sigma$ for some $w$.

(e) Is there a proposition $\delta \in \mathrm{Prop}(A)$ that is satisfied by exactly one truth assignment on $A$?

(f) How do your answers to (d) and (e) change if $A := \{p_i : i \in \mathbb{N}\}$?

2. Recall the function $at : \mathrm{Prop}(A) \to \mathcal{P}(A)$ we defined by:

$$at(x) := x \text{ for all } x \in A;$$

$$at(\neg\alpha) := at(\alpha) \text{ and } at(\wedge\alpha\beta) = at(\vee\alpha\beta) := at(\alpha) \cup at(\beta)$$

for any propositions $\alpha$ and $\beta$.

(a) Write a careful proof that for any proposition $\phi \in \mathrm{Prop}(A)$ and any truth assignments $s$ and $t$ on $A$, if $t(x) = s(x)$ for all $x \in at(\phi)$, then $\bar{t}(\phi) = \bar{s}(\phi)$.

* Suppose that $\phi$ and $\psi$ are propositions, and $\phi$ tautologically implies $\psi$. Show that there exists a proposition $\theta$ with $at(\theta) \subset at(\phi) \cap at(\psi)$ such that $\phi$ tautologically implies $\theta$, and $\theta$ tautologically implies $\psi$; or find a counterexample.

3. Let $\mathrm{Ata}(A)$ be the set of all atomic truth assignments on $A$. Explore the functions $Mod$ and $Th$ we defined in lecture by

$$Mod(\Phi) := \{t \in \mathrm{Ata}(A) : \forall\phi \in \Phi \ \bar{t}(\phi) = T\}$$

and

$$Th(S) := \{\phi \in \mathrm{Prop}(A) : \forall t \in S \ \bar{t}(\phi) = T\}.$$

Describe the notions *tautology*, *satisfiable*, *(tautologically) equivalent*, *model*, and *tautological consequence* (all from p. 16 of our text) in terms of $Mod$ and $Th$.