Math A4400: Mathematical Logic
Problem set 4, due at 2pm on thursday, march 15th.
Solutions turned in after 2:05pm are late and get half credit.

This problem set is about sections 2.3 of Mathematical Logic Lecture Notes by van den Dries.

1. Let S be a signature with a unary function symbol $+$, a binary function symbol 1, a 17-ary relation symbol $<$, and no other symbols.
   (a) Give an example of an S-structure with an infinite universe.
   (b) Give an example of an S-structure with a finite universe.

2. Let S be a signature with one binary function symbol f and no other symbols.
   (a) List the 16 S-structures with universe $\{\mho, \infty\}$.
   (b) Which of the 16 S-structures in part (a) are groups under the interpretation of f? Identify the identity element in each of those.

3. Let S be the signature of rings: two binary function symbols $+$ and $\cdot$, a unary function symbol $-$, and two constant symbols 0 and 1. How many S-structures with universe $\{1, 2, a, X\}$ are there?

4. Consider four signatures $S_i \subset \{P, T, N, Z, W\}$, with P and T binary function symbols, N a unary function symbol, and Z and W constant symbols:

$$S_1 := \{P, T\}, \quad S_2 := \{P, T, N\}, \quad S_3 := \{T, N\}, \quad S_4 := \{P, T, N, W\}.$$

   For each i, let $\mathcal{A}_i$ be the S_i-structure with universe $\mathbb{Z}$ and the following interpretations of symbols in S_i:

$$P^{\mathcal{A}_i}(x, y) = x + y, \quad T^{\mathcal{A}_i}(x, y) = xy, \quad N^{\mathcal{A}_i}(x) := -x,$$

$$Z^{\mathcal{A}_i} := 0, \quad W^{\mathcal{A}_i} := 1.$$

   For each i, describe all S_i-substructures of $\mathcal{A}_i$.

* Let S be a signature with one binary function symbol $*$. Let $\mathcal{A}$ be the S-structure with universe $\mathbb{Z}$ and with $*^{\mathcal{A}}(x, y) := x + y$ for all integers x, y. Let $\mathcal{B}$ be the S-structure with universe $\mathbb{Q}^{>0}$ (the set of all positive rational numbers) and $*^{\mathcal{B}}(x, y) := x \cdot y$ for all rational x, y. Is $\mathcal{A}$ an S-substructure of $\mathcal{B}$? Describe all S-homomorphisms between these two structures.