Math A4400: Mathematical Logic
Problem set 5, due at 2pm on tuesday, march 22nd.
Solutions turned in after 2:05pm are late and get half credit.

0. Make sure that you know and understand all definitions in section 2.3 of Mathematical Logic Lecture Notes by van den Dries.

1. Let S be a signature with one binary relation symbol R and no other symbols, and let $X := \{1, 2, 3, 4, 5\}$. For each of the following S-structures with universe X, list all congruences.

   (a) S-structure $\mathcal{A}$ with universe $A = X$ and $R^{\mathcal{A}} := \{(x, y) \in X^2 : x \leq y\}$.

   (b) S-structure $\mathcal{B}$ with universe $B = X$ and $R^{\mathcal{B}} := \{(x, y) \in X^2 : x \lneq y\}$.

   (c) S-structure $\mathcal{D}$ with universe $D = X$ and $R^{\mathcal{D}} := \{(x, x) : x \in X\}$.

2. Fix a signature S; three S-structures $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$; and two surjective S-homomorphisms $\phi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{A} \to \mathcal{D}$. Suppose that for any $a, a' \in A$, we have $\phi(a) = \phi(a')$ if and only if $\psi(a) = \psi(a')$. Prove or disprove each of the following statements.

   (a) If S contains no relation symbols, then $\mathcal{B}$ is isomorphic to $\mathcal{D}$.

   (b) If S contains no function symbols, then $\mathcal{B}$ is isomorphic to $\mathcal{D}$.

   (c) If ϕ and ψ are strong homomorphisms, then $\mathcal{B}$ is isomorphic to $\mathcal{D}$.

3. Let S be a signature with one binary function symbol $*$. Let $\mathcal{A}$ be the S-structure with universe $\mathbb{Z}$ and with $*^{\mathcal{A}}(x, y) := x + y$ for all integers x, y. Let $\mathcal{B}$ be the S-structure with universe $\mathbb{Q}^{>0}$ (the set of all positive rational numbers) and $*^{\mathcal{B}}(x, y) := x \cdot y$ for all rational x, y. Prove that there is no S-isomorphism between $\mathcal{A}$ and $\mathcal{B}$, or find one.

* Let S and $\mathcal{B}$ be as in Problem 3. Describe the group of S-automorphisms of $\mathcal{B}$.

4. Find a signature S that would be appropriate for vector spaces over the reals. Describe how to turn a vector space V into an S-structure $\mathcal{A}_V$. Are S-substructures of $\mathcal{A}_V$ precisely the subspaces of V? Are S-homomorphisms from $\mathcal{A}_V$ to $\mathcal{A}_W$ precisely the linear transformations from V to W?