**Math A4400: Mathematical Logic**
**Problem set 6, due at 2pm on tuesday, march 29th.**
**Solutions turned in after 2:05pm are late and get half credit.**

**This problem set is about section 2.4 of Mathematical Logic Lecture Notes by van den Dries.**

**The terms that our textbook calls "variable-free" are usually called "closed."**

1. Let $S$ be a signature with one 5-ary function symbol $f$, and let $t$ be the $S$-term $fv_1v_2v_1v_3v_1$. Describe an $S$-structure $\mathcal{A}$ and elements $a, b, c, d \in A$ such that

$$t(v_0, v_1, v_2, v_3)^{\mathcal{A}}(a, b, c, d) = t(v_0, v_1, v_2, v_3)^{\mathcal{A}}(a, c, b, d).$$

2. Fix a signature $S$, two $S$-structures $\mathcal{A}$ and $\mathcal{B}$, an $S$-homomorphism $\phi$ from $\mathcal{A}$ to $\mathcal{B}$, and an $S$-term $t$. Suppose that all variables of $t$ are among $v_0, v_1, \dots v_n$, and that $a_0, a_1, \dots a_n \in A$.
   (a) Work through an example: specify $S$, $\mathcal{A}$, $\mathcal{B}$, and $\phi$; pick some $t$ with at least two function symbols; and use the definition of interpretation of terms to compute the following quantities, writing down every step of the computation.

$$t(v_0, v_1, \dots v_n)^{\mathcal{B}}(\phi(a_0), \phi(a_1), \dots \phi(a_n))$$

$$\phi(t(v_0, v_1, \dots v_n)^{\mathcal{A}}(a_0, a_1, \dots a_n))$$

   (b) Prove that those two quantities are equal for any $S$, $\mathcal{A}$, $\mathcal{B}$, $\phi$, and closed term $t$. Write a careful, detailed proof by induction on terms.
   (c) Prove that for any $S$, $\mathcal{A}$, $\mathcal{B}$, $\phi$, $t$, $n$, and $a_0, a_1, \dots a_n$ as above, the two quantities computed in part (a) are equal. Write a careful, detailed proof by induction on terms.

3. Suppose that $\mathcal{A}$ is an $S$-structure such that for every $a \in A$, there is a closed $S$-term $t$ such that $t^{\mathcal{A}} = a$. Use Problem 2 to show that for any $S$-structure $\mathcal{B}$, there is at most one homomorphism from $\mathcal{A}$ to $\mathcal{B}$.

4. Fix a signature $S$ and an $S$-structure $\mathcal{A}$, and let

$$B := \{t^{\mathcal{A}} : t \text{ is a closed } S\text{-term }\} \subset A.$$

   (a) If $S$ is the signature of rings and $\mathcal{A}$ is your favourite non-commutative ring, what is $B$?
   (b) Show that if $B$ is non-empty, then it is the universe of the smallest $S$-substructure of $\mathcal{A}$.

5. Fix a signature $S$, an $S$-structure $\mathcal{A}$, and elements $a, b \in A$. Suppose that $b$ belong to every $S$-substructure of $\mathcal{A}$ containing $a$. Show that there exists an $S$-term $t$ with at most one variable $v_0$ such that

$$t(v_0)^{\mathcal{A}}(a) = b.$$

* Let $S$ be the signature of ordered abelian groups (see textbook). Describe the $S$-structure $\mathcal{A}$ such that for any $S$-structure $\mathcal{B}$, there is exactly one homomorphism from $\mathcal{A}$ to $\mathcal{B}$.