Math A4400: Mathematical Logic
Problem set 7, due at 2pm on tuesday, april 12th.
Solutions turned in after 2:05pm are late and get half credit.

This problem set is about section 2.5 of Mathematical Logic Lecture Notes by van den Dries.

The terms that our textbook calls "variable-free" are usually called "closed."

1. Let S be the signature with a unary function symbol M, binary function symbols P and T, and a constant symbol W; and let $\mathcal{A}$ be the S-structure with universe $A = \mathbb{R}$, with $W^{\mathcal{A}} := 1$, with $M^{\mathcal{A}}(a) := -a$, and with P interpreted as addition and T interpreted as multiplication.

   (a) Describe the interpretation in $\mathcal{A}$ of the atomic formula

   $$= \quad TPTv_0v_0Mv_2PTv_0v_0Mv_2 \;\; MTPTTv_1v_1v_1Mv_2PTTv_1v_1v_1Mv_2.$$

   (b) Find an atomic S-formula whose interpretation is

   $$\{(t, 1/t) : 0 \neq t \in \mathbb{R}\}.$$

   (c) Find an S-term t such that the image of the term function defined by t is the set in part (a).

   (d) Is there an S-term such that the graph of the term function defined by t is the set in part (b)?

2. Fix a signature S and an atomic S-formula ϕ whose terms only use variables v_0 and v_2. Prove or disprove each of the following.

   (a) For any S-structures $\mathcal{A}$ and $\mathcal{B}$, any S-homomorphism β from $\mathcal{A}$ to $\mathcal{B}$, and any $a, a' \in A$,

   $$\text{if } (a, a') \in \phi(v_0, v_2)^{\mathcal{A}}, \text{ then } (\beta(a), \beta(a')) \in \phi(v_0, v_2)^{\mathcal{B}}.$$

   (b) For any S-structures $\mathcal{A}$ and $\mathcal{B}$, any strong S-homomorphism β from $\mathcal{A}$ to $\mathcal{B}$, and any $a, a' \in A$,

   $$(a, a') \in \phi(v_0, v_2)^{\mathcal{A}} \text{ if and only if } (\beta(a), \beta(a')) \in \phi(v_0, v_2)^{\mathcal{B}}.$$

3. Prove or disprove each of the following statements.

   (a) The graph of any term function is the interpretation of some atomic formula.

   (b) The image of any term function is the interpretation of some atomic formula.

   (c) The preimage of any one point under any term function is the interpretation of some atomic formula.

* For each statement you disproved in Problem 3, find the extra assumption needed to make the statement true.