Math A4400: Mathematical Logic
Problem set 8, due at 2pm on tuesday, april 19th.
Solutions turned in after 2:05pm are late and get half credit.

This problem set is about sections 2.4-2.5 of Mathematical Logic Lecture Notes by van den Dries.

1. Fix a signature S, an S-structures $\mathcal{A}$, and an S-term t whose variables are among v_0, v_1, v_2.
   (a) Show that the function $t(v_0, v_1, v_2)^{\mathcal{A}}$ is the same as the function $u(v_8, v_6, v_7)^{\mathcal{A}}$ where $u = t(v_8/v_0, v_6/v_1, v_7, v_2)$ comes from simultaneous substitution.
   (b) Is $t(v_0, v_1, v_2)^{\mathcal{A}}$ the same function as $w(v_1, v_0, v_2)^{\mathcal{A}}$ where $w = t(v_1/v_0, v_0/v_1)$?
   (c) Is $t(v_0, v_1, v_2)^{\mathcal{A}}$ the same function as $s(v_1, v_0, v_2)^{\mathcal{A}}$ where $s = t(v_1/v_0, v_1/v_2)$?
   (d) Describe the function $\pi : A^5 \to A^3$ such that $t(v_0, v_1, v_2)^{\mathcal{A}} \circ \pi = t(v_0, v_1, v_3, v_2, v_{17})^{\mathcal{A}}$.
2. Fix a signature S and an S-structures $\mathcal{A}$. Suppose that ϕ is an S-formula without quantifiers whose variables are among v_0, v_1, v_2.
   (a) Show that the set $\phi(v_0, v_1, v_2)^{\mathcal{A}}$ is the same as the set $\theta(v_8, v_6, v_7)^{\mathcal{A}}$ where $\theta = \phi(v_8/v_0, v_6/v_1, v_7, v_2)$ comes from simultaneous replacement, defined on p.32.
   (b) Is $\phi(v_0, v_1, v_2)^{\mathcal{A}}$ the same set as $\chi(v_1, v_0, v_2)^{\mathcal{A}}$ where $\chi = \phi(v_1/v_0, v_0/v_1)$?
   (c) Is $\phi(v_0, v_1, v_2)^{\mathcal{A}}$ the same set as $\gamma(v_1, v_0, v_2)^{\mathcal{A}}$ where $\gamma = \phi(v_1/v_0, v_1/v_2)$?
   (d) Show that $\phi(v_0, v_1, v_2)^{\mathcal{A}} \times A^2 = t(v_0, v_1, v_2, v_3, v_{17})^{\mathcal{A}}$.
3. Fix a signature S, an S-structures $\mathcal{A}$, and positive integers m and n. Suppose that $X \subseteq A^m$ and $Y \subseteq A^n$ are definable by S-formulas without quantifiers. Prove that $X \times Y \subseteq A^{m+n}$ is definable by some S-formula without quantifiers; or find a counterexample.
4. Fix a signature S, an S-structures $\mathcal{A}$, a positive integer $m \geq 7$, and a constant symbol $C \in S$. Suppose that $X \subset A^m$ is definable by an S-formula without quantifiers. Prove that the set
   $$\{(a_1, a_3, a_4, \dots a_m) \in A^{m-1} : (a_1, C^{\mathcal{A}}, a_3, a_4, \dots a_m) \in X\}$$
   is definable by an S-formula without quantifiers; or find a counterexample.
5. Let S be the signature with a unary function symbol M, binary function symbols P and T, and a constant symbol W; and let $\mathcal{A}$ be the S-structure with universe $A = \mathbb{Z}$, with $W^{\mathcal{A}} := 1$, with $M^{\mathcal{A}}(a) := -a$, and with P interpreted as addition and T interpreted as multiplication.
   (a) Prove that for any set $X \subset A$ defined by an S-formula without quantifiers, one of X and $A \setminus X$ is finite; or find a counterexample.
   (b) Prove that for any term function $\alpha : A \to A$, one of the image I of α and its complement $A \setminus I$ is finite; or find a counterexample.