Math A4400: Mathematical Logic
Problem set 9, due at 2pm on tuesday, may 3rd.
Solutions turned in after 2:05pm are late and get half credit.

This problem set is about sections 2.4-2.5 of Mathematical Logic Lecture Notes by van den Dries.

1. Let $S := \{P, T\}$ be the signature with two binary function symbols. Consider the following S-structures.

   (a) $\mathcal{A}$ with universe $A := \mathbb{Z}$ and $P^{\mathcal{A}}(a,b) := a+b$ and $T^{\mathcal{A}}(a,b) := ab$

   (b) $\mathcal{B}$ with universe $B := \mathbb{Q}$ and $P^{\mathcal{B}}(a,b) := a+b$ and $T^{\mathcal{B}}(a,b) := ab$

   (c) $\mathcal{C}$ with universe $C := \{\triangle\}$ and $P^{\mathcal{C}}(\triangle,\triangle) := \triangle$ and $T^{\mathcal{C}}(\triangle,\triangle) := \triangle$

   (d) $\mathcal{D}$ with universe D consisting of all 2-by-2 matrices with real entries, and $P^{\mathcal{D}}(a,b) := a+b$ and $T^{\mathcal{D}}(a,b) := ab$

   (e) $\mathcal{E}$ with universe $E := \{e \in \mathbb{Z} : e \geq 17\}$ and $P^{\mathcal{E}}(a,b) := a+b$ and $T^{\mathcal{E}}(a,b) := ab$

   (f) $\mathcal{F}$ with universe $F := \mathbb{R}$ and $P^{\mathcal{F}}(a,b) := a+b$ and $T^{\mathcal{F}}(a,b) := ab$

   (g) $\mathcal{G}$ with universe $G := \{\heartsuit, \diamondsuit\}$ and $P^{\mathcal{G}}(a,b) := \heartsuit$ and $T^{\mathcal{G}}(a,b) := \diamondsuit$ for all a, b

   (h) $\mathcal{H}$ with universe $H := \{\heartsuit, \diamondsuit\}$ and
   $P^{\mathcal{H}}(\heartsuit,\heartsuit) = P^{\mathcal{H}}(\diamondsuit,\diamondsuit) := \heartsuit$ and $P^{\mathcal{H}}(\heartsuit,\diamondsuit) = P^{\mathcal{H}}(\diamondsuit,\heartsuit) := \diamondsuit$ and
   $T^{\mathcal{H}}(\diamondsuit,\diamondsuit) := \diamondsuit$ and $T^{\mathcal{H}}(a,b) = \heartsuit$ for all other (a,b)

   Separate as many of these as you can with as few S-sentences as you can. For example, $\mathcal{C}$ is the only one of them that satisfies $\forall v_0 \, \forall v_1 \; = v_0 v_1$. In principle, it might be possible to separate all 8 structures with only 3 sentences.

2. Let $S := \{R\}$ be the signature with one binary relation symbol. Consider the following S-structures.

   (a) $\mathcal{A}$ with universe $A := \mathbb{N}$ and $R^{\mathcal{A}} := \{(a,b) \in \mathbb{N}^2 : a \leq b\}$

   (b) $\mathcal{B}$ with universe $B := \mathbb{N}$ and $R^{\mathcal{B}} := \{(a,b) \in \mathbb{N}^2 : a \geq b\}$

   (c) $\mathcal{C}$ with universe $C := \mathbb{Q}$ and $R^{\mathcal{C}} := \{(a,b) \in \mathbb{Q}^2 : a \geq b\}$

   (d) $\mathcal{D}$ with universe $D := \mathbb{Q}$ and $R^{\mathcal{D}} := \{(a,b) \in \mathbb{N}^2 : a \geq b\}$

   (e) $\mathcal{E}$ with universe $E := \mathbb{R}$ and $R^{\mathcal{E}} := \{(a,b) \in \mathbb{R}^2 : a \geq b\}$

   (f) $\mathcal{F}$ with universe $F := \mathbb{Z}$ and $R^{\mathcal{F}} := \{(a, a+6) : a \in \mathbb{Z}\}$

   (g) $\mathcal{G}$ with universe $G := \{\heartsuit, \diamondsuit\}$ and $R^{\mathcal{G}} := \{(\heartsuit, \diamondsuit)\}$

   (h) $\mathcal{H}$ with universe $H := \{\heartsuit, \diamondsuit\}$ and $R^{\mathcal{H}} := H^2$

   Separate as many of these as you can with as few S-sentences as you can.

3. Find a good signature S for vector spaces and explain how the symbols of S should be interpreted for a given vector space V to build the corresponding S-structure $\mathcal{V}$. Is the set of pairs of linearly independent vectors definable in your signature? Does the notion of S-substructures and S-homomorphisms correspond to subspaces and linear maps? Is there an S-sentence expressing the property "finite-dimensional"?